

Comment on “Insensitivity of Hawking radiation to an invariant Planck-scale cutoff”

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I point out that the cutoff introduced by Agulló *et al.* [1] has little impact on the trans-Planckian problem as it is usually understood; it excludes only a small fraction of the problematic modes.

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Investigation of Hawking’s [2] suggestion that black holes radiate thermally remains one of the most active areas of research in general relativity. Yet despite an enormous amount of work, two foundational questions remain: the trans-Planckian problem (that is, the dependence of Hawking’s computation on modes of arbitrarily high frequency); and the question of whether the neglect of possible quantum gravitational effects is really correct. (For a review, see Ref. [3].) Indeed, these concerns in part motivate the ingenious attempts to find alternative derivations of Hawking’s results (e.g. [4]).

In a recent interesting paper, Agulló *et al.* [1] argued that Hawking’s computation was in fact insensitive to a certain Lorentz-invariant cutoff, drew parallels with the Unruh effect, and suggested that this might overcome the trans-Planckian problem. I show here that this last hope is not fulfilled: the cutoff removes only a negligible fraction of the relevant trans-Planckian modes.

Much of Agulló *et al.*’s treatment turns on concern over the sense in which the trans-Planckian problem is invariant. In the absence of a well-known compact invariant characterization of this problem, the authors in fact suggest using their manifestly invariant cut-off as the *definition* of the relevant trans-Planckian threshold; were we to accept this, implementing the cut-off would resolve the difficulty. Should we accept this definition? That is, does the cut-off really remove this problematic element of the derivation of Hawking radiation?

To avoid confusion, I will keep the conventional usage of the phrase “trans-Planckian problem,” but I will consider its invariance, as well as the central issue of how much the cut-off helps with this troubling aspect of Hawking’s derivation.

We shall see that the conventional trans-Planckian problem *is* invariant — although a full understanding of this invariance requires non-local considerations. This invariant point of view will also make it clear that only a substantial alteration of Hawking’s original argument could remove the problem; in particular, Agulló *et al.*’s intentionally modest modification cannot resolve it. While this conclusion is negative, we shall find that examining the details of Agulló *et al.*’s argument brings out a great deal of interesting physics in the way these general and non-local considerations manifest themselves

for specific particle detections at particular places.

Following Agulló *et al.*, let us recall the main elements of Hawking’s idea. We consider a linear massless scalar field on a spherically symmetric space-time representing collapse to a black hole of mass M . We resolve the field by spherical harmonics, and consider the propagation of the reduced radial modes. (In fact, it is known that the primary contribution to the Hawking effect is from the s-wave sector, so one could restrict to that.)

The key is to consider what happens to a mode on its passage from the distant past, through the region representing the incipient black hole, and out to the future. A mode whose angular frequency is ω' in the past will give rise to a family of modes in the future, with the Bogoliubov coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ representing the fractional contributions to outgoing modes of angular frequencies $\pm\omega$, respectively. Production of quanta is governed by $\beta_{\omega\omega'}$, and so it is the computation of these Bogoliubov coefficients which is the central element in the analysis. (The angular indices l, m have been omitted.)

Hawking showed that the significant contributions to $\beta_{\omega\omega'}$ arise for $\omega \sim T_H = (8\pi M)^{-1}$ (the Hawking temperature in natural units). However — and this is the trans-Planckian problem — the values of ω' contributing mainly to these, that is, the frequencies of the modes which in the distant past are the precursors to the Hawking quanta, go like

$$\omega' \sim C e^{+u/(4M)}, \quad (1)$$

where u is the retarded time. That is, modes of a fixed frequency $\sim T_H$ in the distant future arise from modes of exponentially increasing frequencies in the distant past. (On the other hand, these modes have, in the distant past, zero occupation numbers. Thus it is vacuum fluctuations of exponentially high frequencies in the distant past which are supposed to be converted, by the Hawking process, into real quanta of moderate frequency in the future.)

I have written, as is conventional, of the modes’ frequencies; of course, measures of frequency presuppose a reference frame, for frequency is not invariant. The frequencies here are all measured with respect to the asymptotic reference frame aligned with the incipient black hole’s energy-momentum. (One could equivalently use the frame of a fixed observer falling freely across the horizon — only a finite relative boost is involved in comparing these frames.) As noted above, much of Agulló *et al.*’s motivation appears to derive from the fact

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that the numerical frequencies of the precursors have, in themselves, no absolute significance. This, while true, gives the impression that the trans-Planckian problem is a frame-dependent one and thus of dubious physical import. However, the problem is an invariant one.

First, the matter in the distant past which will collapse to form the hole defines an approximate reference frame. Attempts to *simultaneously* treat this matter and the Hawking effect by quantum models will involve trans-Planckian wave-vectors whatever frame one works in (and indeed this forms an obstacle to developing a satisfactory theory of the back-reaction of the Hawking effect on the space-time). And second, because the incipient hole is isolated, there is a well-defined way in which we can compare the wave-vectors of the Hawking modes in the distant future with those of their precursors in the distant past (one simply parallel-transport along a path everywhere far from the matter). Thus, while the numerical values of the frequencies are not invariant, the *relative boost* of the precursors' wave-vectors to those of the outgoing quanta is an invariant, and, in a frame in which the outgoing quanta have moderate frequencies the precursors will be trans-Planckian. Another way of saying this is that there is an invariant holonomy got by comparing parallel transport along two paths from the distant past to the future, one the geometric-optics path taken by the Hawking mode, and the other a path everywhere far from the matter [5]. See Fig. 1.

While these trans-Planckian wave-vectors in the distant past correspond to vacuum fluctuations, they can give rise to real physical effects via standard quantum-theoretic principles. Thus real trans-Planckian physics can enter in the consideration of quantum measurement issues involving correlations of detectors in the reference frame in the past with Hawking quanta in the future [6], or, when nonlinear couplings are considered, by altering the detailed balance of the virtual creation and annihilation processes — some of the energy-momentum of these virtual processes can be lost to the incipient hole, resulting in “bubble” diagrams failing to close and the production of real ultra-energetic quanta [7]. The physics in each case is invariant because in each case what matters is the comparison of the reference frame in the past to that in the future via the two different routes giving rise to the holonomy.

But the problem is still more severe, because the wave-vectors of the problematic modes do not merely cross the Planckian threshold; they diverge. The trans-Planckian problem would be manifest *in any fixed* reference frame, for a sequence of wave-vectors which diverges to infinity in one reference frame will do so in any. This applies whether the problematic modes are examined in the distant past, or near the horizon. Thus while the particular frequencies considered are frame-dependent, the trans-Planckian problem is an invariant one, even if we do not wish to appeal to the existence of an approximate physically preferred frame in the distant past, or the non-local holonomy.

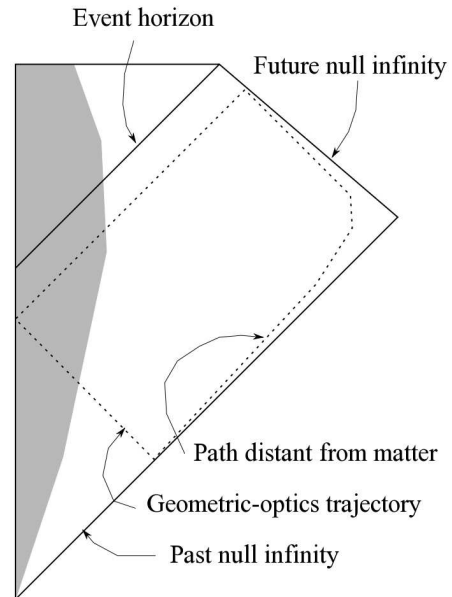


FIG. 1: Conformal diagram of the formation of a spherically symmetric black hole. The matter is shown shaded. The trans-Planckian problem is linked to a holonomy representing the difference in parallel propagation along two paths, one the geometric-optics trajectory of a Hawking mode and the other very distant from the matter. The relative boost given by this holonomy, which carries moderate wave-vectors to trans-Planckian ones, is a physical invariant.

It is worth emphasizing that these arguments about propagation do not establish any overt remarkable behavior of the field itself or the two-point function either near the horizon or in the distant past. What they do show is that *if* one selects the field modes which will, in the future, develop into Hawking quanta, *those* modes behave problematically in these earlier regimes. This is again the statement that the trans-Planckian problem is non-local, since it depends on selecting modes in an early regime on the basis of behavior at a later one. We shall see below how this contrasts with the characterization of trans-Planckian modes proposed by Agulló *et al.*

All of these comments turn on something well-known and, I believe, uncontroversial: that Hawking's original computation of the modes' behavior is mathematically correct, and that, according to it, the Hawking quanta do arise from trans-Planckian precursors. This means that the trans-Planckian problem is very much at the heart of Hawking's original analysis. If we believe Hawking's computation of the Bogoliubov coefficients, then the Hawking quanta do arise from trans-Planckian vacuum fluctuations in the distant past, and no alternative derivation can circumvent this; one would need to suppose different physics applies. (See e.g. [8–11] for such suggestions.)

Agulló *et al.* introduce a cutoff in an invariant way which reproduces Hawking's results (with small cor-

rections). They suggest their cutoff solves the trans-Planckian problem; however, it is evident from the discussion in the previous paragraph that the very nearness of their result to Hawking's means that there can be little change in the contributions of those trans-Planckian modes giving rise to the Hawking quanta. I shall show here that what Agulló *et al.*'s arguments actually establish is that for the Hawking quanta produced in any interval of retarded time Δu a few Hawking-periods long, there is a reference frame in a dimensionally reduced space-time (not the physical space-time itself), relative to which the precursors for the Hawking radiation in Δu are not trans-Planckian. However, there is no single such frame (even in the dimensionally reduced space); as later and later intervals Δu are considered, the corresponding frames are boosted exponentially.

Agulló *et al.* argue as follows. They assert that a detector held just outside the horizon of the incipient black hole will respond in the same way as an Unruh detector of the corresponding acceleration a in Minkowski space. In fact, as will be discussed below, while this assertion is a common one, it is not fully justified. But in fact Agulló *et al.*'s argument does not really depend on such a direct physical correspondence, but rather on the mathematical correspondence between the two-point functions of the two systems.[12] This is given by formulas (8) and (20) in their paper, which are standard.

The authors examine the expected number of Unruh quanta over a proper time interval $\Delta\tau$ with $a\Delta\tau \lesssim 1$. They argue that in the local frame of the detector, the precursors of these Unruh quanta have only moderate frequencies, and that recasting the entire computation in terms of the two-point function $G(x_1, x_2)$ (from which $\beta_{\omega\omega'}$ can be extracted), once can impose the invariant cutoff

$$|G(x_1, x_2)| < l_P^{-2} \quad (2)$$

(where l_P is the Planck length) without substantially affecting the result. They then argue that Eq. (2), carried over into the black-hole case, would give a cutoff theory in which Hawking quanta (slightly modified) were produced but with no trans-Planckian problem.

While Agulló *et al.* are certainly correct that Eq. (2) provides an invariant trans-Planckian cut-off, it is important to recognize that it has, on its face, little to do with the trans-Planckian modes which are problematic for the Hawking process. (Were we to try to formulate the usual trans-Planckian problem in terms of two-point functions, we should expect to use at least two pairs of widely separated points: one pair in the distant past or at the horizon, to detect the precursors' frequencies, and the second pair in the distant future, to detect the Hawking quanta's frequencies.) In order to understand what the significance of restriction (2) is, let us examine the authors' subsequent analysis.

Leaving aside the black-hole case for the moment, Agulló *et al.*'s argument does indeed show that the precursor of an Unruh quantum is, in the frame in which the

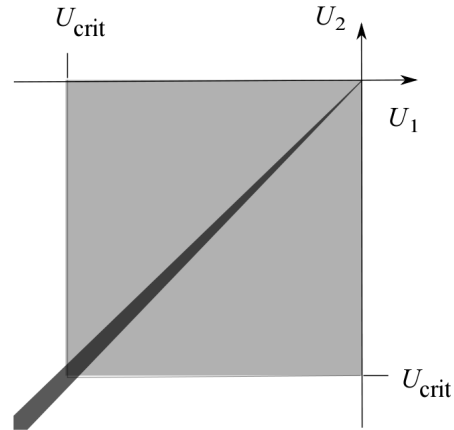


FIG. 2: The part of the U_1 - U_2 plane contributing to the trans-Planckian problem and the sector excluded by the invariant cutoff. The trans-Planckian region is the square $U_{\text{crit}} < U_1, U_2 < 0$; only a narrow sector (shown darkened and with its width exaggerated), of angle $\sim m_P/(32\pi M)$, is excluded by the condition of Agulló *et al.* By contrast, virtually all of the square is required for the modeling of the detection of Hawking quanta.

quantum is detected, of frequency $\sim a$. On the other hand, if one wants to consider the response of the detector over an interval of proper time $\Delta\tau \gg a^{-1}$, then the boosts of the precursors, with respect to a fixed reference frame, over that interval vary by $\sim \exp(a\Delta\tau)$. Thus while over any short time one does not have to invoke ultra-high-frequency modes to explain the Unruh effect, a treatment of it over long times does require exponentially large boosts.

Accepting now the correspondence with the black-hole case, we see that what Agulló *et al.* have shown is that over any short interval Δu of retarded time, there is a frame in which Hawking quanta detected near the horizon have precursors of only moderate frequencies. However, if $\Delta u_1, \Delta u_2$ are two such intervals separated by $\Delta u_{12} = \Delta u_1 - \Delta u_2 \gg 4M$, then the precursors of the Hawking quanta in the intervals are relatively boosted by a factor $\sim \exp \Delta u_{12}/(4M)$.

One should remember, too, that the analyses of detectors set out at different angular positions around the hole will require boosts in different directions, with the boosts of antipodal detectors oppositely directed. Therefore even over a fixed relatively short retarded time interval Δu , the boosts required to find cis-Planckian precursors for the Hawking quanta registering in antipodal detectors will differ from one another by an exponentially increasing factor, $\sim \exp 2u/(4M)$. It is only in the dimensionally reduced problem, where we consider (say) the s-wave sector as a theory in a two-dimensional space, that this issue does not appear.

We can see explicitly that Agulló *et al.*'s cutoff excludes only a small fraction of the relevant trans-Planckian modes by examining the form they actually use in the

black-hole case. This is cast in terms of the Kruskal retarded time

$$U = -(4M) \exp -(u/4M), \quad (3)$$

which is a smooth null coordinate in the vicinity of the horizon.[13] Note that $U < 0$; the limit $U \rightarrow 0$ corresponds to approach to the event horizon.

If we ask when the trans-Planckian problem sets in, we see from relation (1) that there will be a retarded time u , say u_{crit} , beyond which the precursors of the Hawking quanta have trans-Planckian frequencies. This will correspond, according to Eq. (3), to a value $U_{\text{crit}} < 0$ of Kruskal retarded time; the Hawking quanta's precursors will be trans-Planckian for $U_{\text{crit}} < U < 0$.

When we consider the detection of Hawking quanta in the distant future in terms of two-point functions, these can be expressed either using conventional retarded times u_1, u_2 or Kruskal coordinates U_1, U_2 . A detector for quanta of frequency ω will operate over a finite interval of retarded time $\Delta u \gtrsim 1/\omega$; the expected number of quanta it registers will be an integral of the two-point function (weighted by a function encoding the sampling profile of the detector) for u_1, u_2 in the interval. Except for a restricted class of detectors operating early on, this interval will be entirely within the regime $u > u_{\text{crit}}$ corresponding to trans-Planckian precursors. In Kruskal coordinates, for detectors with operating Kruskal times $U > U_{\text{crit}}$, the relevant arguments of the two-point function will be entirely within the trans-Planckian region $U_{\text{crit}} < U_1, U_2 < 0$ of the U_1 - U_2 plane.

The cutoff used by Agulló *et al.* in the black-hole case, their paper's inequality (11), can be written as

$$(U_1 - U_2)^2 > m_{\text{P}}^2 (16\pi M)^{-2} U_1 U_2, \quad (4)$$

where m_{P} is the Planck mass. This cutoff does excise a region from the U_1 - U_2 plane, but it is a small one; a little algebra shows that (for $M \gg m_{\text{P}}$) the condition is equivalent to

$$|U_1/U_2 - 1| > m_{\text{P}}/(16\pi M). \quad (5)$$

Thus only a small fraction $\sim m_{\text{P}}/(32\pi M)$ of the trans-Planckian regime in the U_1 - U_2 plane is excluded. (See Fig. 2.) By contrast, if a detector in the distant future responds over an interval $|u_2 - u_1| \sim n(2\pi/T_{\text{H}}) = n(16\pi^2 M)$ (that is, n times the period of a characteristic Hawking quantum), the corresponding ratio is $U_1/U_2 \sim \exp \pm 4\pi^2 n$ — so nearly the whole square $U_{\text{crit}} < U_1, U_2 < 0$ is required for modeling simply the detection of individual Hawking quanta ($n \sim 1$). Were we to consider correlations between quanta over extended intervals, or the detection of lower-frequency quanta, more of the square would be needed. Thus, of the trans-Planckian regime which actually figures in the Hawking analysis, only a small portion is removed by the condition of Agulló *et al.*

Finally, let me turn to the physical correspondence between the Hawking and Unruh analyses. It is often asserted that a detector held just outside an incipient black hole must, by the equivalence principle, respond like an Unruh detector. While there certainly is a connection, it is not at all such a simple correspondence. A fast way of seeing this is to note that the Unruh analysis is time-symmetric, but Hawking's analysis is definitely not (it uses very strongly that the black hole is formed by collapse; a static black hole would not Hawking-radiate). What actually goes wrong with the correspondence argument is that the equivalence principle applies only locally, but the scales required to compute the Hawking effect are large enough to detect the difference of the quantum state from the Minkowski vacuum. (See Ref. [3] for a quantitative treatment.)

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 - [12] In this connection, one should note that, had they really depended on the physical correspondence, they would have had to account for the ultra-high accelerations of observers hovering just outside the horizon: but Agulló *et al.* consider an Unruh detector with acceleration $a = (4M)^{-1}$.
 - [13] Strictly speaking, the Kruskal coordinate itself is defined only exterior to the matter, but there is a natural extension of it inwards as a null coordinate respecting the spherical symmetry, and this suffices for the analysis; cf. Ref. [3].